ENVELOPE FOLLOWING BASED METHOD FOR THE COMPUTATION OF LYAPUNOV EXPONENTS IN ELECTRONIC CIRCUITS

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ABSTRACT

Lyapunov Exponents (LEs) can be considered as a statistical measure giving some approximate but signifficant insight in the behaviour of a dynamic system. If the systems we consider are models of realistic electronic circuits, we can reasonably assume them to be dissipative. LEs are useful to determine some general topological properties of the attractive region so that, even if the state space has a relatively large dimension, it is possible to foresee the presence of stable equilibrium points, periodic or quasi-periodic trajectories or, possibly, chaos. Lyapunov exponents and related measures such as the Lyapunov dimension are used to characterize the dynamics of complex systems and the geometrical properties of their trajectories in the state space. A few methods are available for the numerical computation of these measures and in most cases they are used only for systems that are normalized, well behaved and with a low dimensional state space. It is here proposed an approach that, being based on the envelope following method for the integration of stiff systems, can be efficiently used for the calculation of Lyapunov exponents and dimension in real electronic circuits.

KEYWORDS: *Circuit Simulation, Autonomous Circuits, Envelope Following, Lyapunov Exponents.*

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INTRODUCTION

Lyapunov Exponents (LEs) can be considered as a statistical measure giving some approximate but signif cant insight in the behaviour of a dynamic system [1], [2]. If the systems we consider are models of realistic electronic circuits, we can reasonably assume them to be dissipative. While this assumption ensures the existence of at least one globally stable region in the state space, it does not give any information concerning the properties of this region. LEs are useful to determine some general topological properties of the attractive region so that, even if the state space has a relatively large dimension, it is possible to foresee the presence of stable equilibrium points, periodic or quasi-periodic trajectories or, possibly, chaos.

Some of these dynamic behaviours can be unforeseen by the circuit designer and, while being chaotic or even hyper-chaotic may be a desirable characteristic for some specif c applications (see e.g. [3] and [4]), in many other cases chaos can have a direct impact on noise, and, in complex circuits having a state space of large dimension where hyper-chaos is a possibility, this undesirable characteristic may even be a source of noise indistinguishable from other physical sources.

Determining LEs is, unfortunately, rather expensive from a computational point of view and is only rarely performed on systems having a large dimensional state space and, possibly, very large scale differences among state variables. An interesting approach that reduces computational cost was proposed for systems with piecewise linear (PWL) nonlinearities [5], [6]. In fact trajectories of PWL systems are a succession of segments of trajectories of linear systems where each of these is determined in closed form so that the only critical problem is to determine the transition time instants between two adjacent linear regions. Since LEs can be easily computed from the system transition matrix and this is immediately available in linear systems a very fast and accurate method is obtained.

In this paper we present a new approach based on the Envelope Following (EF) integration method that, in some way, can be seen as a generalization of the previous PWL method. The EF method was originally proposed to accelerate transient analysis of stiff circuits (see e.g. [7], [8], [9]) and allows very fast convergence in the regions of state space where trajectories behave with some regularity, “slowing down” as necessary in regions where fast variations occur.

In the following we brief y recall some aspects of LEs and the PWL method for their computation. We then outline the EF method used, and, eventually, present some experimental results obtained with our own simulator, implementing the EF method and the here pro-posed LE algorithm.

LYAPUNOV EXPONENTS AND THE PWL METHOD

By assuming that state variables are well def ned for the system, that it is possible to def ne a metric on state space and that trajectories never leave some f nite subset of state space, we can say that, from a geometrical point of view, LEs correspond to a measure of how lengths, surfaces, volumes, ... are stretched or contracted in time by the system vector f eld. As it is well known this corresponds to a measure of how initially close trajectories will tend to become closer or to diverge versus time. All dynamic systems considered in the following are assumed, for simplicity, to be autonomous and to have one single attractive invariant set. This is not a limitation: if more attractors are present we expect the trajectories computed by EF method to remain in the proper attractive basin just as those computed by any other methods.

Consider a continuous time system

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t)) \quad \mathbf{x} \in \mathcal{R}^n \tag{1}$$

given the initial condition $x(t_0) = x_0$ and the corresponding trajectory $\varphi(x_0, t_0, t)$ (or, for ease of notation, $\varphi(t)$), def ne for any $t > t_0$ the corresponding variational equation by introducing a new variable $y(t) = \varphi(t) + \Delta x(t)$ linearizing and neglecting higher order terms we have:

$$\dot{\mathbf{y}}(t) = \left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\boldsymbol{\varphi}(t)} \mathbf{y}(t) \tag{2}$$

The fundamental matrix $\mathbf{\Phi}(t)$ of the system is then def ned as the general solution $y(t) = \mathbf{\Phi}(t)y(0)$ of the variational equation (2). If (1) is periodic of period T then $\mathbf{\Phi}(t + T) = \mathbf{\Phi}(t)\mathbf{\Phi}(T)$ and the eigenvalues ρ_j of $\mathbf{\Phi}(T)$ are the well known Floquet multipliers [2]. In this case LEs are easily def ned as $\lambda_j \equiv (1/T) \log |\rho_j|$. Consider now a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) \tag{3}$$

and its sensitivity to the Δx_0 perturbation of the initial conditions

$$\Delta\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\Delta\mathbf{x}_0 \tag{4}$$

choosing a norm on state space we thus have

$$\frac{\|\Delta\mathbf{x}(t)\|}{\|\Delta\mathbf{x}_0\|} = \|e^{\mathbf{A}(t-t_0)}\tilde{\mathbf{z}}\| \tag{5}$$

for the normalized perturbation vector $\tilde{z} = \Delta x_0/\|\Delta x_0\|$. The right hand side of (5) is bound by a function of the leading eigenvalue of $\mathbf{A}$. In particular, for generic $\tilde{z}$ we have

$$\lim_{t\to\infty} \frac{1}{t - t_0} \log \|e^{\mathbf{A}(t-t_0)}\tilde{\mathbf{z}}\| = \lambda_1 \tag{6}$$

where λ_1 is the leading Lyapunov exponent. This exponent corresponds to a contraction or expansion of lengths whose effects tend to become parallel to its eigenvector. A non generic choice of $\tilde{z}$ is necessary to f nd contractions or expansions along other directions one for each of them.

For a non linear system such as (1) we can assume that small perturbations in the initial conditions still be-

have with an exponential evolution satisfying, in this case, the variational equation (2). We thus have for sufficiently small Δx_0 and again for a generic $\tilde{z}$:

$$\lim_{t\to\infty} \frac{1}{t} \log \|\mathbf{\Phi}(t)\tilde{\mathbf{z}}\| = \lambda_1 \tag{7}$$

and other non leading exponents can be found with specif c values of $\tilde{z}$. Direct computation of (7) for different and non generic values of $\tilde{z}$ is never performed in practice, it is preferable to compute a short step transition matrix that is orthogonalized before the effects of the leading exponent overwhelms those of lesser ones. In the case of PWL systems it is found a closed form expression for the transition matrix $\mathbf{\Phi}(t)$ that can be computed in a fast and accurate way [5], [6].

If an index is assigned to each linear subregion of the PWL system and to the corresponding transition times among these regions, we have

$$\mathbf{\Phi}(t) = e^{\mathbf{A}_k(t-t_k)} e^{\mathbf{A}_{k-1}(t_k-t_{k-1})} \cdots e^{\mathbf{A}_1(t_1-0)} \tag{8}$$

The only critical problem is the determination of transition instants; note that each "partial" transition matrix found in each linear subregion must be orthogonalized to avoid the overwhelming inf uence of the leading exponent.

The method proposed in this paper can be compared in some way to the above approach. EF is used to f nd regions where, even if the system is non linear, the transition matrix can be considered constant within a given tolerance.

THE ENVELOPE FOLLOWING METHOD

Envelope Following methods have been introduced as transient analysis acceleration techniques for highly stiff oscillatory problems, see e.g. [7], [8] and [9]. Their application for the determination of steady states of oscillators has then been presented in [10]. In the sequel the basic idea on which they rely is brief y recalled. Assume that a circuit is excited by a periodic signal of period *T*, denoted as carrier, and that the solution can be represented as an amplitude modulated signal at least in a suitable time interval. We refer to T as the circuit working period. Suppose that modulating signal is at low frequency with respect to the carrier.

By exploiting these assumptions, we may infer that the amplitude of the solution has a small variation from one working period to the subsequent one and then a large number of periods *T* must be considered to have an appreciable relative variation in the amplitude of the modulated signal. This also implies that it is not needed to accurately solve the circuit in each working period but only in a reduced subset where cycles are separated by a time interval multiple of *T*. If this approach is adopted, the circuit may be then simulated by "sampling" the carrier at a low frequency rate, related to the modulating signal.

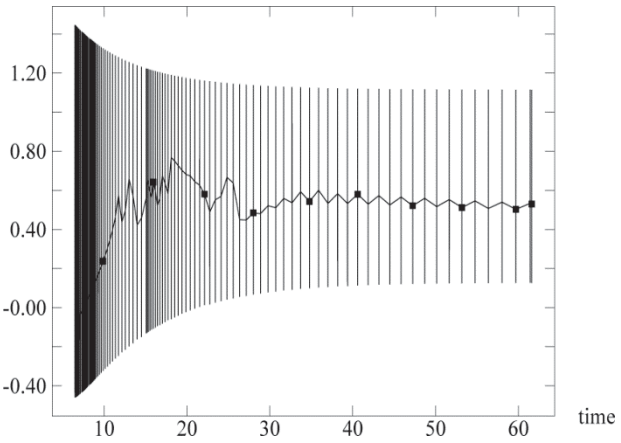

Fig. 1. Sample output waveform for the LC-tuned bipolar oscillator in Figure 2 evaluated using the EF algorithm. Time is in μs

The obtained samples constitute a discrete version of a waveform that is referred to as envelope. It is straightforward to note that the EF approach reduces the number of working periods simulated and speeds up time domain simulation. The acceleration effect is immediately visible in Figure 1, where the output voltage of the LC-tuned oscillator later used in numerical experiments is shown. From a different point of view the EF method performs a dynamic linearization of the non linear system adjusting the linearization each time a given tolerance is not satisf ed. Note that the PWL approach previously outlined exploits the intrinsic characteristics of the system model, so that its transition matrix is def ned as the ordered product of constant and well def ned matrices, each scaled by an exponent that is proportional to the time interval in which the system remains in the corresponding region.

Even if the system model is not PWL the EF method tries to f nd regions where the transition matrix can be considered as constant within a given tolerance. At the beginning of each sample the transition matrix $\mathbf{\Phi}(T)$ corresponding to one period of length T (or approximated period if, as in general, the system is non periodic) is computed and a predictor − corrector algorithm is then used to evaluate what is the maximum time step as a multiple of the period, i.e. nT , so that the transition matrix can simply be taken to be $\mathbf{\Phi}_n(T)$. To this end consider once more (8), the corresponding equation as evaluated by the EF algorithm is thus:

$$\mathbf{\Phi}(t) = \mathbf{\Phi}_k^{n_k}(T_k)\mathbf{\Phi}_{k-1}^{n_{k-1}}(T_{k-1}) \cdots \mathbf{\Phi}_1^{n_1}(T_1) \tag{9}$$

with

$$t = \sum_{j=1}^{k} n_j T_j$$

and where, for the j-th step of EF, T_j is the approx-imated period, $\mathbf{\Phi}_j$ (T_j) the corresponding transition matrix, and nj a scalar integer such that n_j T_j is the maximum time interval in which the corresponding transition matrix can be

considered constant. If the system is periodic and is in its steady state $T_j = T$ ∀j and all partial transition matrices will be equal. In this case (9) reduces to

$$\mathbf{\Phi}(t) = \mathbf{\Phi}(nT) = \mathbf{\Phi}_1^n(T) \tag{10}$$

In our implementation we have chosen to reduce numerical problems induced by positive leading exponents by using the simple **QR** factorization method (see e.g. [1]), denoting as $\boldsymbol{Q}_j\boldsymbol{R}_j$ the factorization per-formed at the *j*-th step, i.e. for the term $\mathbf{\Phi}_j^{nj}(T_j)\boldsymbol{Q}_{j-1}$, we have, at the $(j + 1)$th step

$$\mathbf{\Phi}(t) = \underbrace{\mathbf{\Phi}_{j+1}^{n_{j+1}}(T_{j+1})\mathbf{Q}_j}_{\mathbf{Q}_{j+1}\mathbf{R}_{j+1}}\mathbf{R}_j\mathbf{R}_{j-1}\cdots\mathbf{R}_1 \tag{11}$$

where the factorization that is performed in the cur-rent step has been evidenced. Since all R matrices are triangular, eigenvalues are immediately found as term by term products.

As an effect of the approximation introduced by EF it is interesting to note that convergence to LEs is less noisy than other methods. This is not a surprise since EF is an averaging method and tends to suppress high frequency components.

EXPERIMENTAL RESULTS

The algorithm presented in the previous sections has been implemented in our simulator [11] as an option of the EF analysis. Since EF def nes a sort of dynamic PWL approximation of a non linear system, we f rst compare our results to those obtained from the well known Chua circuit that, being PWL ab origine, is a signif cant benchmark for our method. This circuit and the related normalization schemes are well know in literature (see [12]).

Results have been f rst compared to those outlined in [5] and [6] for the normalized Chua circuit using the reference PWL method. As it can be seen in Table I, results obtained with the two methods closely agree. The control parameter is α, and a '*' has been appended in cases corresponding to chaotic behaviour; results agree both when the dynamic behaviour is periodic or chaotic.

Table I

Lyapunov Exponents (LEs) for the normalized Chua, PWL and EF methods compared

Normalized Chua circuit						
α	PWL method			EF method		
8.0	0	-0.12	-1.22	0.0002	-0.1205	-1.2241
8.5*	0.08	0	-1.43	0.0688	0.0000	-1.4230
9.0*	0.23	0	-1.75	0.2328	-0.0112	-1.7320
9.78*	0.26	0	-2.26	0.2594	-0.0027	-2.1014
10.05	0	-0.02	-1.61	0.0001	-0.0620	-1.5695
10.5*	0.3	0	-2.25	0.2944	-0.0110	-2.1903

Results for a non-normalized Chua circuit are presented in Table II; as expected LEs have in this case a relevant scale change if compared to the corresponding normalized case (while preserving relative proportions). For example the null exponent is far from zero in absolute value, but not if compared to other exponents. In fact LEs are invariant for coordinate change, but are affected by changes in time scale (and, as expected, if these LEs are multiplied by the corresponding time scaling coefficient, in this case $\approx 7.0 \cdot 10^{-5}$, the normalized exponents are once more found). In Table II we also report the Lyapunov dimension (LD) obtained using the formula in [13], i.e. as a function of the relative values of the exponents. This measures, def ned only for strange attractors, gives some information about geometric properties of the attractor; it is also invariant for time scale changes so that, as shown in Table II, its value does not vary in the normalized or non normalized case.

Table II

Lyapunov exponents (LEs) and dimension (LD) for the non normalized Chua circuit

Non Normalized Chua circuit, EF method					
α	LEs			LD	LD (norm)
8.0	-3.1	-1728.0	-17498.5	—	—
8.5*	1183.4	-26.5	-20553.2	2.05628	2.04822
9.0*	3295.25	-144.0	-25056.7	2.1258	2.1279
10.5*	4363.39	-55.1267	-31673.7	2.13602	2.1294

An application of the method to a realistic circuit has been tested on the LC-tuned oscillator shown in Figure 2 along with circuit parameter values. The circuit has a periodic steady state behaviour as shown in Figure 1. Since all parasitic components have been included in the BJT model the state space has a larger dimension than expected by considering the transistor simply as a non linear resistive device.

Results for this circuit are summarized in Table III, the oscillator working frequency is 308.54 MHz but results have been normalized to 1 Hz.

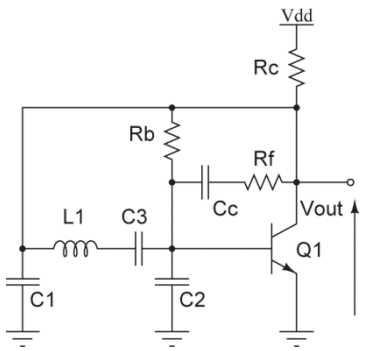

Fig. 2. The circuit of the LC-tuned bipolar oscillator: C1 = 33 pF, C2 = 33 pF, C3 = 3.17 pF, Cc = 560 pF, L1 = 100 nH, Rf = 680Ω, Rb = 100 K Ω, Rc = 1.2 K Ω

Table III

Lyapunov Exponents of the LC-tuned oscillator in Figure 2, state space has dimension 8 for the presence of 3 parasitic capacitances in the BJT model

LC−tuned oscillator, LEs			
0.0002	-0.0267	-0.208	-0.208
-3.27	-3.31	-3.38	-4.09

As a last example consider the Colpitts oscillator in Figure 3, the crystal is tuned at 5 MHz and has a quality factor Q = 50000. Since, as for the previous circuit, 3 parasitic capacitors are included in the BJT model and 3 state variables are contributed by the crystal, the state space for this circuit has a total dimension of 9.

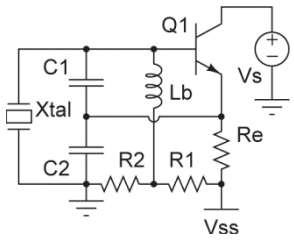

Fig. 3. The circuit of the Colpitts oscillator: C_1 = 33 pF, C_2 = 560 pF, L_b = 390 μH, R_1 = 33 KΩ, R_2 = 22 *K*Ω, R_e = 1.5 *K*Ω, V_s = 10 V, V_{ss} = -10 V

Also in this case the results reported in Table IV have been normalized using the oscillator frequency as scaling factor.

Table IV

Lyapunov Exponents of the Colpitts oscillator

Cristall Colpitts oscillator, LEs				
0.0002	-0.0145	-0.9426	-5.7829	-6.9743
-7.0267	-7.0575	-7.0571	-7.3513	

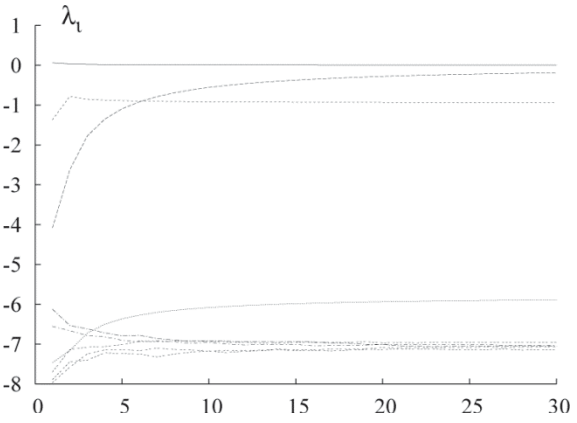

Fig. 4. Lyapunov Exponents of the Colpitts oscillator in Figure 3 converge in few iteration of the EF algorithm

Convergence of LEs using the proposed method is quite fast and smooth, in Figure 4 we show the first 30 iterations of the EF method. Results are within few percents of the final values already after few iterations.

CONCLUSIONS

A fast method for the determination of the complete Lyapunov spectrum of electronic circuits has been presented. The method has the advantage of computing LEs during the normal circuit simulation (with a small computational overhead). Due to averaging properties of the EF method, upon which it is based, convergence is fast and affected by a small level of numerical noise.

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