# A MIXED TIME-FREQUENCY DOMAIN APPROACH FOR THE QUALITATIVE ANALYSIS OF AN HYSTERETIC OSCILLATOR

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**ABSTRACT**

Frequency domain techniques, like harmonic balance and describing function, are classical methods for studying and designing electronic oscillators and nonlinear microwave circuits. In most applications spectral techniques have been used for determining the steady-state behavior of nonlinear circuits that exhibit a single periodic attractor. On the other hand, the global dynamics of nonlinear networks and systems is usually investigated through time-domain techniques, that require to introduce rather complex and sophisticated concepts. Recently some HB based techniques have been proposed for investigating bifurcation processes in nonlinear circuits that present several attractors (the authors have considered systems that admits of a Lur'e representation). Their approach presents the advantages of providing a simple and qualitative description of the system dynamics, that can be effectively exploited for design purposes. In this manuscript we will examine a third order hysteretic oscillator, that cannot be described in the classical Lur'e form and we will show that its dynamics can be investigated through the joint application of the describing function technique and of a suitable time-domain method for estimating Floquet's multipliers.

**KEYWORDS:** *Hysteretic Oscillator, Frequency domain techniques, Floquet's multipliers.*

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Frequency domain techniques, like harmonic balance and describing function, are classical methods for studying and designing electronic oscillators and nonlinear microwave circuits [1], [2], [3]. In most applications spectral techniques have been used for determining the steady-state behavior of nonlinear circuits that exhibit a single periodic attractor. On the other hand, the global dynamics of nonlinear networks and systems is usually investigated through time-domain techniques, that require to introduce rather complex and sophisticated concepts [4].

Recently some HB based techniques have been proposed for investigating bifurcation processes in nonlinear circuits that present several attractors [5]-[11]. In [12], [13] the authors have considered systems that admits of a Lur'e representation. They have shown that the describing function technique (i.e., HB with a single harmonic) is able to predict the occurrence of chaotic behavior and several bifurcation phenomena. Their approach presents the advantages of providing a simple and qualitative description of the system dynamics, that can be effectively exploited for design purposes.

In this manuscript we will examine a third order hysteretic oscillator, that cannot be described in the classical Lur'e form and we will show that its dynamics can be investigated through the joint application of the describing function technique and of a suitable time-domain method for estimating Floquet's multipliers.

We consider the third order hysteretic oscillator shown in Figure 1 of [14] and described by the following set of normalized state equations:

$$\dot{x} = -(x+y) \quad (1)$$

$$\dot{y} = \alpha(x+y) - y - \beta z \quad (2)$$

$$\dot{z} = \gamma f(x-z) - \delta \sinh(z) \quad (3)$$

where the dot denotes the time-derivative and α, β δ and γ depend on circuit parameters. According to [14] f(·) is defined as the R → R function, obtained by finding for each w the unique solution of the following transcendental equation.

$$f = \delta \sinh[w - f] \quad (4)$$

As a preliminary step we show that the above set of equations can be reduced to a scalar Lur'e like system. The first two equations (1) and (2) allow one to derive a linear relationship between $x(t)$ and $z(t)$:

$$z(t) = L(D)x(t) \quad (5)$$

where $D = d/dt$ denotes the first order differential operator and

$$L(D) = \frac{D^2 + (2-\alpha)D + 1}{\beta} \quad (6)$$

By substituting expression (5) in equation (3), we obtain the following Lur'e like model, in term of the scalar variable x(t):

$$DL(D)x(t) = \gamma f[x(t) - L(D)x(t)] - \delta \sinh[L(D)x(t)] \quad (7)$$

As pointed out above, we will show that the most significant dynamic properties of the hysteretic oscillator under study, can be qualitatively revealed through the joint application of the describing function technique and of a suitable time-domain method for computing limit cycle Floquet's multipliers.

We assume that any periodic signal of period $T = 2\pi/\omega$ can be approximated by a bias term and a single harmonic. We have

$$x(t) \approx \tilde{x}(t) = A + B\sin(\omega t) \quad (8)$$

where $A$ denotes the bias term and $B$ is the amplitude of the first order harmonic.

The following expressions can be readily computed:

$$L(D)\tilde{x}(t) = L(0)\,A + B\,\mathbf{Re}[L(j\omega)]\,\sin(\omega t) + B\,\mathbf{Im}[L(j\omega)]\,\cos(\omega t) \quad (9)$$

$$DL(D)\tilde{x}(t) = B\,\omega\,\mathbf{Re}[L(j\omega)]\,\cos(\omega t) - B\,\omega\,\mathbf{Im}[L(j\omega)]\,\sin(\omega t) \quad (10)$$

$$L(0) = \frac{1}{\beta},\ \mathbf{Re}[L(j\omega)] = \frac{1-\omega^2}{\beta},\ \mathbf{Im}[L(j\omega)] = \frac{(2-\alpha)\omega}{\beta} \quad (11)$$

By exploiting (11), it is also derived that expression $f[\tilde{x}(t) - L(D)\tilde{x}(t)]$ can be considered as a function (hereafter named $\tilde{F}(\cdot)$) of the describing function parameters $A$, $B$ and ω and of time:

$$f[\tilde{x}(t) - L(D)\tilde{x}(t)] = f\left[\left(1-\frac{1}{\beta}\right)A + B\left(1-\frac{1-\omega^2}{\beta}\right) \times \sin(\omega t) - B\,\frac{(2-\alpha)\omega}{\beta}\cos(\omega t)\right] = \tilde{F}(A,B,\omega,t) \quad (12)$$

According to the describing function technique, the following first harmonic approximation of function $f[\tilde{x}(t) - L(D)\tilde{x}(t)]$ holds:

$$f[\tilde{x}(t) - L(D)\tilde{x}(t) \approx F^A(A,B,\omega) + F^B(A,B,\omega)\sin(\omega t) + F^C(A,B,\omega)\cos(\omega t) \quad (13)$$

where

$$F^A(A,B,\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{F}(A,B,\omega,t)d\omega t$$

$$F^B(A,B,\omega) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{F}(A,B,\omega,t)\sin(\omega t)d\omega t$$

$$F^C(A,B,\omega) = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{F}(A,B,\omega,t)\cos(\omega t)d\omega t \quad (14)$$

Since the explicit expression of function $f(\cdot)$ is not known (because the latter is found as the solution of the tran-scendental equation (4)), the integrals above do not admit analytical expressions.

However they can be numerically computed to any desired accuracy, for any set of parameters *A*, *B* and ω.

By following a similar procedure, the first harmonic approximation of $\sinh[L(D)x(t)]$ (the other nonlinear function appearing in the scalar model (7)) can be found. We have:

$$\sinh[L(D)\tilde{x}(t)] \approx G^A(A,B,\omega) + \\ + G^B(A,B,\omega)\sin(\omega t) + G^C(A,B,\omega)\cos(\omega t) \quad (15)$$

where

$$\begin{aligned} G^A(A,B,\omega) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh[L(D)\tilde{x}(t)]d\omega t \\ G^B(A,B,\omega) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh[L(D)\tilde{x}(t)]\sin(\omega t)d\omega t \\ G^C(A,B,\omega) &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh[L(D)\tilde{x}(t)]\cos(\omega t)d\omega t \end{aligned} \quad (16)$$

The integral expressions (16) can be analytically computed. In fact, after some algebraic manipulations, the following fundamental integrals can be derived (for any set of real coefficients *R P*, *Q*).

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sinh[R + P\sin(\theta) + Q\cos(\theta)]\,d\theta = \\ = I_0(\sqrt{P^2+Q^2})\,\sinh(R) \quad (17)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh[R + P\sin(\theta) + Q\cos(\theta)]\,\sin(\theta)\,d\theta = \\ = 2\frac{P}{\sqrt{P^2+Q^2}}I_1(\sqrt{P^2+Q^2})\,\cosh(R) \quad (18)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sinh[R + P\sin(\theta) + Q\cos(\theta)]\,\cos(\theta)\,d\theta = \\ = 2\frac{Q}{\sqrt{P^2+Q^2}}I_1(\sqrt{P^2+Q^2})\,\cosh(R) \quad (19)$$

where $I_0$ and $I_1$ denotes the modified Bessel function of the first kind of order zero and one respectively.

Since, according to (9) and (11) the following expression stands for $L(D)\tilde{x}(t)$

$$L(D)\tilde{x}(t) = \frac{1}{\beta}A + \frac{1-\omega^2}{\beta}B\,\sin(\omega t) + \\ + \frac{(2-\alpha)\omega}{\beta}B\,\cos(\omega t) \quad (20)$$

the analytical close form for the integrals (16) can be readily derived by replacing *R*, *P* and *Q* in (17) - (19) with the expressions shown below:

$$R = \frac{1}{\beta}A, \quad P = \frac{1-\omega^2}{\beta}B, \; Q = \frac{(2-\alpha)\omega}{\beta}B \quad (21)$$

By substituting in (7) the first order harmonic approximations of the nonlinear functions $f[\tilde{x}(t) - L(D)\tilde{x}(t)]$ and $\sinh[L(D)\tilde{x}(t)]$ derived in (13) and (15), and of $DL(D)\tilde{x}(t)$ derived in (10), we obtain a non-di˙erential system of three equations with three unknowns *A*, *B* and ω (hereafter named describing function system):

$$\begin{aligned} \gamma F^A(A,B,\omega) - \delta G^A(A,B,\omega) = 0 \\ B\,\omega\,\mathbf{Im}[L(j\omega)] + \gamma F^B(A,B,\omega) - \delta G^B(A,B,\omega) = 0 \\ B\,\omega\,\mathbf{Re}[L(j\omega)] - \gamma F^C(A,B,\omega) + \delta G^C(A,B,\omega) = 0 \end{aligned} \quad (22)$$

where Re[$L(j\omega)$] and Im[$L(j\omega)$] are given by (11) and the coefficient $F^{A,B,C}$ and $G^{A,B,C}$ have the expressions reported in (14) and (16) respectively.

The describing function system can be solved in a very efficient way, by exploiting standard numerical methods. If for a given set of parameters α, β, γ and δ a solution (*A*, *B* and ω) exists, the latter is called predicted limit cycle. Since the describing function technique is approximate in nature, in general one cannot guarantee that there is a one to one correspondence between the describing function solutions (predicted limit cycles) and the actual limit cycles of the system. There are however two possible approaches for discussing the accuracy of the describing function prediction. The first one consists in the computation of the distortion index Δ, that for the system under study can be defined by adapting the well known definition given in [13] for classical Lur'e systems:

$$\mathbf{\Delta}(A,B,\omega) = \frac{||\overline{x}(t) - \tilde{x}(t)||_2}{||\tilde{x}(t)||_2} \quad (23)$$

where

$$\overline{x}(t) = [DL(D)]^{-1}\{\gamma f[\tilde{x}(t) - \\ - L(D)\tilde{x}(t)] - \delta\sinh[L(D)\tilde{x}(t)]\} \quad (24)$$

If the distortion index is small enough, i.e. $\Delta < 0.01$, then the describing function prediction can be considered reliable.

The second approach requires to verify a set of rather complex conditions, under which one can guarantee the existence of an actual limit cycle in a computable neighborhood of the predicted limit cycle [3].

As we have already pointed out, several describing function based methods have been developed for investigating limit cycle stability properties and their most significant bifurcation phenomena [13]. Such methods mainly rely on the Loeb's stability criterion and are based on the investigation of the homogeneous linearized system equation, obtained by perturbing the original solution in a way that depends on the bifurcation under study.

In this manuscript we will show that an accurate investigation of limit cycle stability and bifurcations, can be carried out by linearizing the system equation (1)-(3) along the solution predicted by the describing function technique and then by computing the Floquet's multipliers of the corresponding variational equation. By denoting with $\tilde{w}(t) = [\tilde{x}(t), \tilde{y}(t), \tilde{z}(t)]'$ the describing function solution and with $w_p(t)$ a generic perturbation of $\tilde{w}(t)$, the variational equation is readily derived:

$$\dot{w}_p(t) = A(t)w_p(t) \tag{25}$$

$$A(t) = \begin{pmatrix} -1 & -1 & 0 \\ \alpha & \alpha - 1 & -\beta \\ \gamma f'(\tilde{x} - \tilde{z}) & 0 & -\gamma f'(\tilde{x} - \tilde{z}) - \delta \cosh(\tilde{z}) \end{pmatrix} \tag{26}$$

where $\tilde{z}(t)$ can be computed as $L(D)[\tilde{x}(t)]$ by exploiting (5) and (9); $f'(\cdot)$ denotes the derivative of function $f(\cdot)$ with respect to its argument, whose expression can be obtained from (4):

$$f'(w) = \frac{\delta \cosh[w - f(w)]}{1 + \delta \cosh[w - f(w)]} \tag{27}$$

Once matrix $A(t)$ is known, the Floquet's multipliers can be effectively computed by using the time-domain numerical algorithm described in [15].

The application of the mixed time-frequency domain technique described above, allows one to identify and characterize the main dynamic features of the hysteretic oscillator under study. We will show in the final version of the manuscript that the most significant bifurcation curves (pitchfork and flip bifurcations) can be detected and that the parameter regions in which more attractors coexist can be identified with a good accuracy (see Fig. 2 of [14], where a bifurcation brute-force analysis was carried out, for a synthetic description of the oscillator dynamic behavior). The accuracy of the technique has been checked by comparing the results with those obtained by applying the spectral method described in [17], based on the approximation of the state through a large number of harmonics.

As a conclusion we remark the main characteristics of our approach, in comparison with previous works on spectral techniques [13] and on hysteretic oscillators [14].

*Remark 1:* The proposed techniques applies to an hysteretic oscillator, that cannot be described as a classical Lur'e system. Then it represents a sort of extension of the general results presented in [13].

*Remark 2:* In comparison with the spectral technique described in [13] the proposed method represents an alternative and simpler way for studying bifurcations, that in some cases gives rise to more accurate results (see for example [16], where bifurcations in a Colpitts' oscillator were investigated). Moreover in all cases that we have considered the predictions yielded by our method are never less accurate and precise than those provided by applying the technique developed in [13].

*Remark 3:* With respect to the brute force method presented in [14], the proposed technique allows one to predict the existence of unstable limit cycles, that play an important role for understanding bifurcation phenomena. Hence it provides a more complete knowledge of the global dynamics of the hysteretic oscillators.


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