# THE CONTROLLABILITY OF RLCM NETWORKS OVER F(Z) AND THEIR APPLICATIONS

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**ABSTRACT**

The subject of controllability and observability in dynamical systems is one of the important matters in linear system theory. The controllability and observability in linear systems over the field R of real numbers were heavily studied. If the values of resistors, capacitors, and self and mutual inductors in RLCM network are regard as independently variable parameters, the network is defined over the field F(z) of all rational functions in its physical parameters. Some concepts and results on the reducibility and separability for this network are presented; the controllability problem and application are discussed. In this paper the author will discuss the relationship between separability and reducibility of RLCM networks over F(z), the relationship between the physical structure of RLCM networks and structural controllability. The results in this paper can be used to study different kinds of linear or nonlinear electrical networks (over F(z)).

**KEYWORDS:** *RLCM networks over F(z), Controllability, Reducibility, Separability.*

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## Introduction

The subject of controllability and observability in dynamical systems is one of the important matters in linear system theory. The controllability and observability in linear systems over the field R of real numbers were heavily studied. However, uncontrollability (unobservability) is a "singular" condition in the sense that if the system $\dot{X}$ =AX+Be, Y=CX+De is uncontrollable (unobservable), then almost any small perturbation of the elements of A and B (A and C) will cause it to become controllable (observable) [1]. Why does it happen? When the effect of the values of elements in A and B (A and C) is excluded? What is the independent of effect of its structure? In order to analyze the issue and explore the independent of system structures and structural controllability, many researchers proposed various parametrizations [2-11].

Let F(z) denote the field of all rational functions with real coefficients in q independently variable parameters $z_1,\ldots, z_q$. Let $z=(z_1 ,\ldots, z_q)$. The domain of z is $R^q$ and $R^q$ is also said to be the parameter space. Let F(z)[s] denote the ring of all F(z)-coefficient polynomials in s. A matrix M is called a rational function matrix (RFM) or a matrix over F(z) if each entry of the matrix is a member of F(z). A linear system is a rational function system (RFS) or a system over F(z) if all coefficient matrices A, B, C and D of the system are RFMs. For example, if all physical parameters (resistors, capacitors, self and mutual inductors) of an RLCM network-a network consisting of resistors, capacitors, and self and mutual inductors, are considered to be mutually independently variable parameters, then all coefficient matrices of the network are RFMs and it is an RFS or an RLCM network over F(z).

The field R of real numbers is a subfield of F(z) and the controllability over F(z) is equivalent to structural controllability (see Section 2). Thus, the controllability of the systems over F(z), that is, structural controllability, is more universal. The systems over R can be a class of systems over F(z).

For a physical system with given structure such as an RLCM network with given topology, its coefficient matrices A, B, C and D are ones over R and it is a system over R only if its parameter values (ie., the values of its resistors, capacitors, and self and mutual inductors) are given. So, the controllability and observability of the system over R depend on both its physical structure and parameter values. If all physical parameters of the system are regarded as mutually independently variable parameters, not as real constants, then it is a system over F(z). The controllability of the system over F(z) is independent of the values of its physical parameters because the values of *z* have never been given for the controllability of the system over F(z). So the effect of the parameter values is excluded, and only that of the physical structure is left. Therefore the controllability over F(z) is just the structural controllability of the physical system.

In this paper the author will discuss the relationship between separability and reducibility of RLCM networks over F(z), the relationship between the physical structure of RLCM networks and structural controllability. Example 7.1 in Section 7 shows that analyzing the controllability of RLCM systems over F(z) may be significant from the viewpoint of physics. The results in this paper can be used to study different kinds of linear or nonlinear electrical networks (over F(z)).

## Preliminaries

Consider a linear system over F(z)

$$\dot{X} = AX+Be,\quad Y=CX+De, \tag{2-1}$$

Where A,B,C and D are, respectively, n×n, n×m, p×n, and p×m matrices, $X \in R^n$, $U \in R^g$, $Y \in R^e$ and RES(2-1) has q variable parameters $z=(z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_q) \in R^q$. Let T=[sI-A,B] and $T_0=[sI-A', C']'$. Let

$M=\{z \quad R^q \mid \text{rank}T=n, \ \forall s \quad \hat{C}\}$ or
$M=\{z\in R^q \mid \det(T\overline{T})\neq 0, \ \forall s\in\hat{C}\}$,
$N=\{z\in R^q \mid \text{rank}T_0=n, \ \forall s\in\hat{C}\}$ or
$N=\{z\in R^q \mid \det(T_0\overline{T_0})\neq 0, \forall s\in\hat{C}\}$,

where $\hat{C}$ denotes complex field, $T$ and $\overline{T_0}$ are complex conjugate of T and $T_0$ respectively. Let S be a point set and m*S denote the Lebesgue measure of the set S.

**Definition 1.** RES(2-1) is structurally controllable (SC) if m*M=1; otherwise, it is not SC. RES(2-1) is structurally observable (SO) if m*N=1; otherwise, it is not SO.

**Definition 2.** RES(2-1) is controllable over F(z) or it is controllable for short if $\det(T\overline{T})$ is nonzero member of F(z) for $\forall$ s $\in \hat{C}$; otherwise, it is uncontrollable. RES(2-1) is observable over F(z) or it is observable for short if $\det(T_0\overline{T_0})$ is a nonzero member of F(z) for $\forall$ s $\in \hat{C}$; otherwise, it is unobservable.

**Lemma 1.** Let f(z) $\in$ F(z). If f(z) is a zero member of F(z) (simply f(z)=0), then f(z)=0 for all z$\in R^q$; if f(z) is a nonzero member of F(z) (simply f(z)$\neq$ 0), $m^*\{z\in R^q \mid f(z)\neq 0\}=1$.

**Remark 1.** By the above Definitions and Lemma 1. SC(SO) is equivalent to the controllability

Let A be an n×n matrix over F(z), A is said to be reducible under $P_1AP_1^{-1}$ or simply to be reducible if there exists some nonsingular matrix $P_1$ over F(z) such that

$$P_1AP_1^{-1}=\begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}, \qquad (2\text{-}2)$$

where $A_i$ is an $n_i\times n_i$ matrix, i=1,2, $1\leqslant n_i<n$; otherwise A is irreducible. A is said to be reducible under PA $P'$ if there exists some permutation matrix P such that

$$PAP'=\begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix}, \qquad (2\text{-}3)$$

where $A_i$ is an $n_i\times n_i$ matrix, i=1,2, $1\leqslant n_i<n$; otherwise, A is irreducible under PA $P'$.

**Reducibility condition**

**Corollary 3.1**: Let K$\subseteq$F be field, $A=(V+W)^{-1}T$, where $V=\text{diag}[\zeta_1, \ldots ,\zeta_n]$ is a matrix over F such that $\zeta_1, \ldots ,\zeta_n$ are algebraically independent over the field K, W with indeterminates $\alpha_1,\ldots,\alpha_{m1}$ and T with indeterminates $\beta_1,\ldots,\beta_{m2}$ are two n×n matrices over K, $z=(\zeta_1, \ldots ,\zeta_n, \alpha_1,\ldots,\alpha_{m1}, \beta_1,\ldots,\beta_{m2})$, and T is invertible, then the following propositions are equivalent.

1. A is an n×n irreducible matrix over F(z) (or det (λI-A) is irreducible).
2. A is an n×n irreducible matrix under PA $P'$ over F(z).

*Proof*: This proof is obvious by Theorem 3 and Corollary 3 in [11].

**Corollary 3.2**: Let K$\subseteq$F be field, $A=(V+W)^{-1}T$, where $V=\text{diag}[\zeta_1, \ldots ,\zeta_n]$ is a matrix over F such that $\zeta_1, \ldots ,\zeta_n$ are algebraically independent over the field K, W and T are two n×n matrices over K, then there exist some permutation P such that A is reducible under PA $P'$ iff there exist some permutation $P_1$ such that H=[s(V+W)-T] is reducible under $P_1HP_1'$, where s is a parameter independent of $\zeta_1, \ldots ,\zeta_n$.

*Proof:* Since $A=(V+W)^{-1}T$, then by Theorem 3 in [11], A is irreducible under PA $P'$ iff [V+W-Tt] is irreducible under $P_1$[V+W-Tt] $P_1'$. Moreover H=[s(V+W)-T]= s(V+W-Tt), $t=s^{-1}$, the reducibility of H is equivalent to the reducibility of (V+W-Tt). Hence, this corollary is true.

**Controllability (observability) Criteria**

**Theorem 4.1**: A is an n×n irreducible matrix over F(z), the equation (2-1) is controllable (observable) over F(z) iff B 0 (C≠0)

*Proof*: we shall first proof the controllability.

*Necessity:* it is obvious that if B=0, then (2-1) is uncontrollable

*Sufficiency:* since A is irreducible and B 0, then rank[sI-A, B]=n at every eigenvalue of A. if not, then there exist an eigenvalue λ and a 1×n vector α 0 such that

$$\alpha[\lambda I-A,B]=0 \quad \text{or} \quad \alpha\lambda=\alpha A \text{ and } \alpha B=0$$

which imply $\alpha A^2=\lambda\alpha A=\lambda^2\alpha$

and in general, $\alpha A^i=\lambda^i\alpha$ i=1,2,…

Let $T=[B \ AB \ \ldots \ A^{n-1}B]$ is the controllability matrix, hence we have

$$\alpha T=[\alpha B \ \lambda\alpha B \ \ldots \ \lambda^{n-1}\alpha B]=0$$

which means rankT=k<n.. Then by Theorem 1 in [12], there exists some n×n nonsingular matrix P over F(z) such that

$$[\bar{A},\bar{B}]=[P^{-1}AP,P^{-1}B]=\begin{pmatrix} A_c & 0 & 0 \\ A_{21} & A_c & B_c \end{pmatrix}$$

which is contradicts the fact that A is irreducible. So rank[sI-A, B]=n. This implies that (A,B) is controllable, that is, the equation (2-1) is controllable over F(z).

the equation (2-1) is observable over F(z) iff C 0, because the observability and controllability are coupled.

**Relationship between Separability and Reducibility**

**Definition 3.** A graph is said to be separable (hinged) if there is at least one subgraph that has at most one node in common with its complement subgraph in the graph; otherwise it is an unhinged graph. A subgraph in a hinged graph is called an unhinged subgraph if it has at most one node in common with its complement subgraph and it is unhinged. Clearly, there are k unhinged subgraphs in a hinged graph, k⩾2. An RLCM network is separable if its graph is hinged; otherwise, it is unhinged. A subnetwork in a hinged network is unhinged if its graph is an unhinged subgraph. An RLCM network is electrically separable if ① its graph is hinged, there is no mutual coupling between any branch of one unhinged subnetwork and any branch of another unhinged subnetwork; otherwise, it is electrically unhinged.

In order to discuss relationship between separability and reducibility, we should first systematically formulate the state equation of an RLCM network. According to Electrical Network Theory[13], we write KVL equation for the fundamental-loop and KCL equation for the f-cut-set in matrix form:

$$\begin{bmatrix} V_{Cl} \\ V_{Rl} \\ V_{Ll} \\ V_J \end{bmatrix} = \begin{bmatrix} Q'_{EC} & Q'_{CC} & 0 & 0 \\ Q'_{ER} & Q'_{CR} & Q'_{RR} & 0 \\ Q'_{EL} & Q'_{CL} & Q'_{RL} & Q'_{LL} \\ Q'_{EJ} & Q'_{CJ} & Q'_{RJ} & Q'_{LJ} \end{bmatrix} \begin{bmatrix} V_E \\ V_{Ct} \\ V_{Rt} \\ V_{Lt} \end{bmatrix},$$

$$\begin{bmatrix} I_E \\ I_{Ct} \\ I_{Rt} \\ I_{Lt} \end{bmatrix} + \begin{bmatrix} Q_{EC} & Q_{ER} & Q_{EL} & Q_{EJ} \\ Q_{CC} & Q_{CR} & Q_{CL} & Q_{CJ} \\ 0 & Q_{RR} & Q_{RL} & Q_{RJ} \\ 0 & 0 & Q_{LL} & Q_{LJ} \end{bmatrix} \begin{bmatrix} I_{Cl} \\ I_{Rl} \\ I_{Ll} \\ I_J \end{bmatrix} = 0 \qquad (5\text{-}1)$$

where all variables in capital letter are in Laplace -transformed form. $V_{Ct}(I_{Ct})$, $V_{Rt}(I_{Rt})$, $V_{Lt}(I_{Lt})$, $V_{Cl}(I_{Cl})$, $V_{Rl}(I_{Rl})$, $V_{Ll}(I_{Ll})$, $V_E(I_E)$ and $V_J(I_J)$ are $n_{Ct}$, $n_{Rt}$, $n_{Lt}$ $n_{Cl}$, $n_{Rl}$, $n_{Ll}$, $n_E$, and $n_J$ dimentional vector. each element of $V_{Ct}(I_{Ct})$, $V_{Rt}(I_{Rt})$, $V_{Lt}(I_{Lt})$, $V_{Cl}(I_{Cl})$, $V_{Rl}(I_{Rl})$ and $V_{Ll}(I_{Ll})$ corresponds to each diagonal element of $C_t$, $R_t$, $L_t$, $C_l$, $R_l$ and $L_l$, for example, $V_{Ct}=[V_{Ct}(1),\ldots,V_{Ct}(n_{Ct})]'$, $C_t=\mathrm{diag}[C_t(1),\cdots,C_t(n_{Ct})]$, $V_{Ct}(i)$ is the voltage across $C_t(i)$, $i=1,\cdots,n_{Ct}$.

For RLCM network, we have following voltage-current relations:

$$\begin{bmatrix} I_{Rt} \\ I_{Rl} \end{bmatrix} = \begin{bmatrix} G_t & 0 \\ 0 & G_l \end{bmatrix} \begin{bmatrix} V_{Rt} \\ V_{Rl} \end{bmatrix}, \begin{bmatrix} V_{Lt} \\ V_{Ll} \end{bmatrix} = \begin{bmatrix} sL_t & sM_{tl} \\ sM_{lt} & sL_l \end{bmatrix} \begin{bmatrix} I_{Lt} \\ I_{Ll} \end{bmatrix} = sL \begin{bmatrix} I_{Lt} \\ I_{Ll} \end{bmatrix},$$

$$\begin{bmatrix} I_{Ct} \\ I_{Cl} \end{bmatrix} = \begin{bmatrix} sC_t & 0 \\ 0 & sC_l \end{bmatrix} \begin{bmatrix} V_{Ct} \\ V_{Cl} \end{bmatrix} \qquad (5\text{-}2)$$

In above expressions $G_l$, $G_t$, $C_l$ and $C_t$ are the matrices of conductance links and twigs, capacitor links and twigs; they are diagonal. Because of the possibility of mutual inductance, there may be coupling between inductor twigs and links, which means $M_{tl}\neq 0$, as well as between inductor twigs themselves and inductor links themselves, which means $M_{ll}$ and $M_{tt}$ are not zero. So the inductor matrices $L$ are not diagonal, but $L_l=L_{ll}+M_{ll}$ and $L_t=L_{tt}+M_{tt}$ are symmetric, $L_{ll}$ and $L_{tt}$ are diagonal, $M_{ll}=M'_{ll}$, $M_{tt}=M'_{tt}$, $M_{tl}=M'_{lt}$.

From (5-1) and (5-2), we have

$$\begin{bmatrix} s(C_t+Q_{CC}C_lQ'_{CC})+Q_{CR}R^{-1}Q'_{CR} & Q_{CL}-Q_{CR}R^{-1}Q'_{RR}R_tQ_{RL} \\ Q'_{CL}-Q'_{RL}G^{-1}Q_{RR}G_lQ'_{CR} & -s(L_l+Q'_{LL}L_tQ_{LL}-Q'_{LL}M'_{lt} \\ & -M_{lt}Q_{LL})-Q'_{RL}G^{-1}Q_{RL} \end{bmatrix} \begin{bmatrix} V_{Ct} \\ I_{Ll} \end{bmatrix}$$

$$= \begin{bmatrix} -Q_{CR}R^{-1}Q'_{ER}-sQ_{CC}C_lQ'_{EC} & Q_{CR}G_lQ'_{RR}G^{-1}Q_{RJ}-Q_{CJ} \\ Q'_{RL}G^{-1}Q_{RR}G_lQ'_{ER}-Q'_{EL} & Q'_{RL}G^{-1}Q_{RJ}+s(Q'_{LL}L_tQ_{LJ} \\ & -M_{lt}Q_{LJ}) \end{bmatrix} \begin{bmatrix} V_E \\ I_J \end{bmatrix} \qquad (5\text{-}3)$$

where

$G=G_t+Q_{RR}G_lQ'_{RR}$, $R=R_l+Q_{RR}R_tQ'_{RR}$, $G_t^{-1}=R_t$, $G_l^{-1}=R_l$, $Q'_{CL}-Q'_{RL}G^{-1}Q_{RR}G_lQ'_{CR}=(Q_{CL}-Q_{CR}R^{-1}Q'_{RR}R_tQ_{RL})'$

Suppose that there are no loops containing just capacitors and independent voltage source and there are no cut-sets containing just inductors and independent current sources, which means that $Q_{CC}sC_lQ'_{EC}=0$ and $Q'_{LL}L_tQ_{LJ}-M_{lt}Q_{LJ}=0$. To simplify, define

$$E=\begin{bmatrix} -Q_{CR}R^{-1}Q'_{ER} & Q_{CR}G_lQ'_{RR}G^{-1}Q_{RJ}-Q_{CJ} \\ -Q'_{RL}G^{-1}Q_{RR}G_lQ'_{ER}+Q'_{EL} & -Q'_{RL}G^{-1}Q_{RJ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

$$V=\begin{bmatrix} C_t & 0 \\ 0 & L_l \end{bmatrix}, \quad W=\begin{bmatrix} Q_{CC}C_lQ'_{CC} & 0 \\ 0 & Q'_{LL}L_tQ_{LL}- \\ & Q'_{LL}M'_{lt}-M_{lt}Q_{LL} \end{bmatrix},$$

$$T=\begin{bmatrix} -Q_{CR}R^{-1}Q'_{CR} & -Q_{CL}+Q_{CR}R^{-1}Q'_{RR}R_tQ_{RL} \\ Q'_{CL}-Q'_{RL}G^{-1}Q_{RR}G_lQ'_{CR} & -Q'_{RL}G^{-1}Q_{RL} \end{bmatrix}, \qquad (5\text{-}3a)$$

then (5-3) can be written as

$$[s(V+W)-T][V'_{Ct}\ \ I'_{Ll}]'=E[V'_E\ \ I'_J]' \qquad (5\text{-}4)$$

According to Electrical Network Theory, if an RLCM network contains no perfectly coupled inductors, the principal minors of $L$ will be nonsingular, which implies that $(V+W)^{-1}$ exists, so take the inverse Laplace transform of (5-4), we have

$$\frac{d}{dt}\begin{bmatrix} v_{Ct} \\ i_{Ll} \end{bmatrix} = (V+W)^{-1}T\begin{bmatrix} v_{Ct} \\ i_{Ll} \end{bmatrix} + (V+W)^{-1}E\begin{bmatrix} v_E \\ i_J \end{bmatrix} \qquad (5\text{-}5)$$

This is a matrix differential equation of the first order. Equation (5-5) can be written in compact matrix notation as

$$dx/dt = Ax + Be \qquad (5\text{-}6)$$

where the meaning of the matrices of A, B, $x$ and $e$ are obvious. Since all the resistors, capacitors, and self and mutual inductors are regarded as mutually independently variable parameters, so from (5-3a), we know that matrix A in (5-6) is similar to matrix A in Corollary 3.1, so we have following conclusion:

**Theorem 5.1**: RLCM network (5-6), when shorting voltage-sources and opening current-source, ie., $e$=0, is electrically unhinged iff there does not exist any permutation matrix P such that A is irreducible under PA $P'$.

*Proof*: Obviously, A is reducible under PA $P'$ if the network (5-6) without sources is electrically hinged (separapable). It is only necessary to prove the necessity.

Since A=$(V+W)^{-1}$T, then by Corollary 3.2, if A is reducible under PA $P'$, there must exist some permutation matrix $P_1$ such that H= s(V+W)-T is reducible under $P_1$H $P_1'$, that is

$$P_1HP_1' = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} = P_1\begin{bmatrix} \Psi & \vartheta \\ -\vartheta' & \Phi + s\mathcal{M} \end{bmatrix}P_1' \qquad (5\text{-}7)$$

where $H_1$ is an $\gamma\times\gamma$ matrix, $1\le\gamma<n_{Ct}+n_{Ll}$; all $\Psi$, $\vartheta$, $\Phi$ and $\mathcal{M}$ are not zero,

$$\Psi = s(C_t + Q_{CC}C_l Q_{CC}') + Q_{CR}R^{-1}Q_{CR}',$$
$$\vartheta = Q_{CL} - Q_{CR}G_l Q_{RR}' G^{-1}Q_{RL},$$
$$\Phi = s(L_{ll} + Q_{LL}' L_{tt}Q_{LL}) + Q_{RL}' G^{-1}Q_{RL},$$
$$\mathcal{M} = (M_{ll} + Q_{LL}' M_{tt}Q_{LL} - Q_{LL}' M_{lt}' - M_{lt}Q_{LL}).$$

Clearly, $P_1$H $P_1'$ will have three kinds of expressions as follows

$$① \begin{bmatrix} \Psi_1 & 0 & 0 \\ 0 & \Psi_2 & \vartheta \\ 0 & -\vartheta' & \Phi + s\mathcal{M} \end{bmatrix}, \quad ② \begin{bmatrix} \Psi & \vartheta & 0 \\ -\vartheta' & \Phi_1 + s\mathcal{M}_1 & 0 \\ 0 & 0 & \Phi_2 + s\mathcal{M}_2 \end{bmatrix},$$

$$③ \begin{bmatrix} \Psi_1 & \vartheta_1 & & 0 \\ -\vartheta_1' & \Phi_1 + s\mathcal{M}_1 & & \\ & & \Psi_2 & \vartheta_2 \\ 0 & & -\vartheta_2' & \Phi_2 + s\mathcal{M}_2 \end{bmatrix},$$

where $\Psi_i$, $\vartheta_i$, $\Phi_i$ and $\mathcal{M}_i$ are submatrices of $\Psi$, $\vartheta$, $\Phi$ and $\mathcal{M}$, i=1,2. If $\mathcal{M}$=0, which means there are no mutual coupling among the inductor branches, RLCM network will change into RLC network, then by Theorem 3.1 in [14], the network is hinged, there exist at least two unhinged subnetworks. If $\mathcal{M}$ 0, RLCM network will also be hinged because mutual inductance do not influence separability of the network; from above three expressions, we know that there exist at least two subnetworks $G_1$ relating with matrix $H_1$ and $G_2$ relating with matrix $H_2$, where $G_1$ and $G_2$ are unconnected or they have one node in common, expression shows that there are no electrical mutual coupling between $G_1$ and $G_2$ because $G_1$ has no inductor, expression and expression show the same result because there are no mutual inductance between the inductor branches of $G_1$ and the inductor branches of $G_2$ (which is caused by the reducibility of $\mathcal{M}$: there exist some permutation $P_2$ such that $P_2\mathcal{M}P_2'$=diag( $\mathcal{M}_1, \mathcal{M}_2$ ) ). So if A is reducible under PA $P'$, then RLCM network is electrically unhinged.

**Some results of Controllability**

In (5-6), let $\dot{X}$ =$dx/dt$, $X$=$x$, then we have

$$\dot{X} = AX + Be \qquad (6\text{-}1)$$

where A=$(V+W)^{-1}$T, B=$(V+W)^{-1}$E, (6-1) is same as the first equation of (2-1). By Electrical Network Theory, if the network without source does not contain any independent all-inductor loops or independent all-capacitor cut-sets, T in (5-3a) will be invertible, so according to Theorem 5.1, Corollary 3.1 and Theorem 4.1, we have following result.

**Result 1:** an electrically unhinged RLCM network (6-1), in which ① there are no loops containing just capacitors and independent voltage source and there are no cut-sets containing just inductors and independent current sources ② there is no independent all-inductor loop or independent all-capacitor cut-set when shorting voltage-sources and opening current-source, is controllable and observable over F(z), i.e., structurally controllable, iff B≠0.

Now let us turn to when B is not zero. B=$(V+W)^{-1}$E, since $(V+W)^{-1}$ is nonsingular matrices, so B≠0 iff E≠0.

By Electrical Network Theory, we know that if there only exist independent voltage sources in the network, we can get the coefficient submatrices $E_{11}$, $E_{21}$ in the following ways: open all reactive links and short all reactive twigs, then $E_{11}$ is computed by finding the currents $i_{11}$ in the shorted capacitor twigs and $E_{21}$ is computed by finding the voltage $v_{21}$ at the open inductor links, it is impossible that $i_{11}$ and $v_{21}$ are zero simultaneously, so if $v_E \neq 0$, then $E_1=[E'_{11} \quad E'_{21}]' \neq 0$. In a similar way, we have: if $i_J \neq 0$, then $E_2=[E'_{12} \quad E'_{22}]' \neq 0$, this means that if there exist the sources in the network, E must not be zero, that is $B \neq 0$. An result is easy to yield as follows

**Result 2:** an electrically unhinged RLCM network without perfectly coupling inductors, in which ① there are no loops containing just capacitors and independent voltage source and there are no cut-sets containing just inductors and independent current sources ② there is no independent all-inductor loop or independent all-capacitor cut-set when shorting voltage-sources and opening current-source, is always controllable over F(z), i.e., is structurally controllable.

### Applications to RLCM networks

**Example 7.1.** Consider an RLCM network without perfectly coupling as shown in Fig.1.

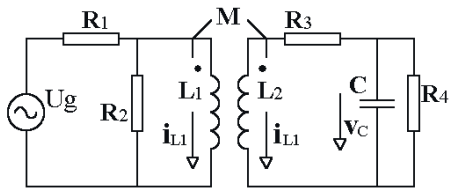


Fig. 1. an RLCM network

Let the state vector be $X=(v_C, i_{L1}, i_{L2})'$ and the input be the voltage source Ug. The state equation of the network is $\dot{X}=AX+Bu$, where

$$A=\frac{1}{\Delta}\begin{bmatrix} \frac{M^2-L_1L_2}{R_4} & 0 & M^2-L_1L_2 \\ -CM & -CL_2(R_1//R_2) & -CMR_3 \\ CL_1 & CM(R_1//R_2) & CL_1R_3 \end{bmatrix}, B=\frac{1}{\Delta}\begin{bmatrix} 0 \\ CL_2 \\ -CM \end{bmatrix}$$

$\Delta=C(L_1L_2-M^2)$, $R_1//R_2=R_1R_2/(R_1+R_2)$, $L_1L_2-M^2>0$ (7-1)

Let C, $R_1$, $R_2$, $R_3$, $R_4$, $L_1$, $L_2$, M be eight independently variable parameters (not be eight real constants), i.e., z=( C, $R_1$, $R_2$, $R_3$, $R_4$, $L_1$, $L_2$, M). Then, the network described by (7.1) is not one over R but one over F(z). Look at the physical structure of Fig.1., the network is hinged but is electrically unhinged, so by Result 2, the network is controllable over F(z), i.e., $m^*S=\{z \in R^5 \mid \text{rank}[sI-A,B]=3, \forall s \in \hat{C}\}=1$ by Definition 1, obviously, this is dependent only on the network structure.

If $R_1=R_2=2\,\Omega$, $R_3=R_4=1\Omega$, C=1F, $L_1$=2H, $L_2$=M=1H, then

$$A=\begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & -1 \\ 2 & 1 & 2 \end{bmatrix}, B=\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad (7\text{-}2)$$

are two matrices over R and the network described by (7-2) is one over R, hence we have

$$[sI-A,B]=\begin{bmatrix} s+1 & 0 & 1 & 0 \\ 1 & s+1 & 1 & 1 \\ -2 & -1 & s-2 & -1 \end{bmatrix},$$

when s=0, then rank[sI-A,B]=2<3, so the network (7-2) over R is uncontrollable, clearly, the result is caused by both the physical parameter values (C, $R_1$, $R_2$, $R_3$, $R_4$, $L_1$, $L_2$, M)= (1, 1, 1, 1, 1, 2, 1, 1) and the structure of the network, because (7-2) is determined by both. $m^*S=\{z\in R^5 \mid \text{rank}[sI-A,B]=3, \forall s\in\hat{C}\}=1$ means that the point set $\bar{S}=(z\in R^5 \mid \text{rank}[sI-A,B]<2)$ is a hypersurface in parameter space $R^8$, it is impossible that a point in parameter space $R^8$ fall on hypersurface $\bar{S}$ accurately. So when $z=\bar{z}=(1, 1, 1, 1, 1, 2, 1, 1)\in\bar{S}$, then $m^*\bar{S}=0$, which implies that if the values of $z$ are at random chosen in the parameter space $R^8$, then the probability that the network over $R$ is controllable is one. This shows that if a network over F(z) is controllable, i.e., structurally controllable, then actually its network over R is always controllable . Therefore, analyzing the controllability of linear physical systems over F(z) is significant from the practical viewpoint.


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